Abelian Projection of $SU(2)$ Gluodynamics and Monopole Condensate [*]

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ABSTRACT

The general properties of the abelian projection are reviewed. We derive the explicit expression for the abelian functional integral for $U(1)$ abelian theory which corresponds to the abelian projection of the $SU(2)$ gluodynamics. The numerical results for the temperature dependence of the monopole condensate are presented.

1 Introduction

The monopole mechanism [1] of the colour confinement is generally accepted by the lattice community. Still there are many open questions. In order to discuss the abelian monopoles in the vacuum of gluodynamics we have to perform the abelian projection [2]. Some general properties of abelian projection are presented in Section 2. To simplify the formulae we discuss the Maximal Abelian (MaA) projection of the $SU(2)$ gluodynamics. The generalization to the $SU(N)$ ($N > 2$) gauge theory and to other abelian projections is straightforward. To prove that the vacuum of gluodynamics behave as the dual superconductor we have to show that in the confinement phase there exists the monopole condensate. This is a nonperturbative problem and we perform the computer simulations of the lattice gluodynamics. To study the monopole condensate we need the explicit expression for the operator $\Phi_{mon}(x)$, which creates the abelian

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monopole at the point x. The operator $\Phi_{mon}(x)$ was found for the compact electrodynamics with the Villain form of the action by Fröhlich and Marchetti [3], and it was studied numerically in Refs.[4]. The generalization of the operator Φ_{mon} to the abelian projection of the gluodynamics is suggested in Refs. [5]. The vacuum expectation value of $< \Phi_{mon} >$ defines the value of the monopole condensate. In Section 3 we describe the numerical results for the temperature dependence for thus defined monopole condensate. It occurs that at the low temperature (confinement phase) $< \Phi_{mon} > \neq 0$ and at the high temperature (deconfinement phase) $< \Phi_{mon} > = 0$. This result shows that in the confinement phase the vacuum of the gluodynamics is similar to the dual superconductor.

2   Maximal Abelian Projection

The partition function of $SU(2)$ gauge theory is:

$$\mathcal{Z} = \int \mathcal{D}A \, \exp\left\{ -\frac{1}{4} \int \mathrm{d}^4x \, F_{\mu\nu}^2[A] \right\}, \tag{1}$$

where $F_{\mu\nu}$ is the field strength tensor: $F_{\mu\nu}^a[A] = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g\,\epsilon^{abc} A_\mu^b A_\nu^c$.

The MaA gauge is defined as the maximization of the functional

$$R[A] = -\int \mathrm{d}^4x \left(\left(A_\mu^1\right)^2 + \left(A_\mu^2\right)^2 \right), \tag{2}$$

over the gauge transformations, $A_\mu \to A_\mu^\Omega = \Omega^+ A_\mu \Omega - \frac{i}{g} \Omega^+ \partial_\mu \Omega$. The functional R is invariant under the $U(1)$ subgroup of the $SU(2)$ gauge group, therefore the maximization condition fixes $SU(2)$ gauge freedom up to $U(1)$ gauge group.

The gauge fixing procedure is standard. We define the Faddeev–Popov unity:

$$1 = \Delta_{FP}[A; \lambda] \cdot \int \mathcal{D}\Omega \, \exp\{\lambda R[A^\Omega]\}, \quad \lambda \to +\infty, \tag{3}$$

where Δ_{FP} is the Faddeev–Popov determinant. We substitute the unity (3) in the partition function (1), shift the fields by the regular transformation Ω^+: $A \to A^{\Omega^+}$ and use the gauge invariance of the Haar measure, the action and the Faddeev-Popov determinant under the regular gauge transformations. Thus we get the product of the volume of the gauge orbit, $\int \mathcal{D}\Omega$, and the partition function in the fixed gauge:

$$\mathcal{Z}_{MaA} = \int \mathcal{D}A \, \exp\left\{ -\int \mathrm{d}^4x \left[\frac{1}{4} F_{\mu\nu}^2[A] - \lambda\left(\left(A_\mu^1\right)^2 + \left(A_\mu^2\right)^2 \right) \right] \right\} \Delta_{FP}[A; \lambda]. \tag{4}$$

In the non–degenerate case the FP determinant can be represented in the form:

$$\Delta_{FP}[A; \lambda] = Det^{\frac{1}{2}} M[A^{\Omega_r^{MaA}}] \, \exp\left\{ -\lambda R[A^{\Omega_r^{MaA}}] \right\} + \ldots, \tag{5}$$

where $\Omega_r^{MaA} = \Omega_r^{MaA}(A)$ is the regular gauge transformation which corresponds to a global maximum of the functional $R[A^\Omega]$, the dots correspond to the terms which are suppressed in the limit $\lambda \to \infty$; and

$$M_{xy}^{ab}[A] = \left.\frac{\partial^2 R(A^{\Omega(\omega)})}{\partial\omega^a(x)\,\partial\omega^b(y)}\right|_{\omega=0}, \tag{6}$$

$\Omega(\omega) = \exp\{i\omega^a T^a\}$, $T^a = \sigma/2$ are the generators of the gauge group, σ^a are the Pauli matrices. In the limit $\lambda \to +\infty$ the region of the integration over the fields A reduces to region where the gauge fixing functional R is maximal, and therefore the partition function (4) can be rewritten as follows [6]:

$$\mathcal{Z}_{MaA} = \int \mathcal{D}A \exp\{-S(A)\} Det^{\frac{1}{2}}\Big(M[A]\Big)\Gamma_{FMR}[A], \tag{7}$$

where $\Gamma_{FMR}[A]$ is a characteristic function of the Fundamental Modular Region [7] for the MaA projection[6].

After the abelian projection we get an abelian theory which contains two dynamical variables, namely, abelian gauge fields $\mathcal{A}$ and monopole currents j. The origin of singular monopole currents will be discussed later. The explicit calculation shows [8] that these currents form closed loops in the four-dimensional space.

The abelian gauge field $\mathcal{A}$ is the component of the $SU(2)$ gauge field which transforms as an abelian gauge field under the residual $U(1)$ gauge transformations in the MaA gauge: $\mathcal{A}_\mu = A_\mu^3$. The $SU(2)/U(1)$ invariant definition of the abelian fields is: $\mathcal{A}_\mu = (A_\mu^{\Omega_r^{MaA}(A)})^3$.

The abelian monopole trajectory j corresponding to the given gauge field configuration A is defined as follows. We maximize the gauge fixing functional $R[A^\Omega]$ with respect to both regular and singular gauge transformations Ω. The definition of the abelian monopole current is:

$$j_\mu(A) = \frac{g}{2\pi}\epsilon_{\mu\nu\alpha\beta}\partial_\nu f_{\alpha\beta}((A^{\Omega_s^{MaA}(A)})^3). \tag{8}$$

where $f_{\mu\nu}$ is the abelian field strength tensor $f_{\mu\nu}(A) = \partial_\mu A_\nu - \partial_\nu A_\mu$. The monopole currents (8) are conserved and quantized [8], and they can be represented as follows:

$$j_\mu(x) = \int_0^T ds\, \frac{\partial\bar{x}_\mu(s)}{\partial s}\delta^{(4)}(x - \bar{x}(s)), \tag{9}$$

the four–vector $\bar{x}_\mu(s)$, $s \in [0, T]$ parameterizes the closed monopole trajectory, $\bar{x}_\mu(0) = \bar{x}_\mu(T)$.

The field configuration A contains the abelian monopoles if the maximizing transformation Ω_s^{MaA} is singular. The non–abelian field strength tensor transforms under the singular gauge transformations as follows:

3

$$
\begin{aligned}
F_{\mu\nu}[A] \rightarrow F_{\mu\nu}[A^{(\Omega)}] &= \Omega^+ F_{\mu\nu}[A]\Omega + F_{\mu\nu}^{sing}[\Omega]\,, \\
F_{\mu\nu}^{sing}[\Omega] &= -i\Omega^+(x)[\partial_\mu\partial_\nu - \partial_\nu\partial_\mu]\Omega(x)\,.
\end{aligned}
\tag{10}
$$

The abelian field strength tensor $f_{\mu\nu}$ in the MaA projection can be decomposed into two parts, $f_{\mu\nu} = f_{\mu\nu}^r + f_{\mu\nu}^s$, where $f_{\mu\nu}^r$ is the regular part, $\epsilon_{\mu\nu\alpha\beta}\partial_\nu f_{\alpha\beta}^r = 0$, and $f_{\mu\nu}^s$ is the singular part, $f_{\mu\nu}^s(A) = F_{\mu\nu}^{sing,3}[A^{\Omega_s^{MaA}}]$. Thus the monopoles appear due to the singularities of the abelian fields strength tensor.

The partition function (4) can be represented as the functional integral over the abelian gauge field and over the abelian monopole trajectories. The abelian monopoles in the MaA projection in the functional integral formalism can be treated [9] similarly to the t'Hooft–Polyakov monopoles [10] in the Georgi–Glashow model.

We define the unity

$$
\begin{aligned}
1 &= \Delta_{ab}[A] \cdot \sum_{n=0}^{+\infty} \frac{1}{n!} \int \prod_{i=1}^{n} \int \mathcal{D}\bar{x}^{(i)} \int \mathcal{D}\mathcal{A} \cdot \\
&\quad \delta\Big(j_\mu - \frac{g}{2\pi}\epsilon_{\mu\nu\alpha\beta}\partial_\nu f_{\alpha\beta}\big((A^{\Omega_s^{MaA}(A)})^3\big)\Big) \cdot \delta\Big(\mathcal{A}_\mu - (A_\mu^{\Omega_r^{MaA}(A)})^3\Big)\,,
\end{aligned}
\tag{11}
$$

where $\Delta_{ab}[A]$ is a "Jacobian" for the change of variables $A \rightarrow \{\mathcal{A}, j\}$ and the parameters $\bar{x}_\mu^{(i)}$ correspond to the disconnected parts of the monopole trajectories. Due to the closeness of the currents j the measure $\mathcal{D}\bar{x}$ includes the integration $\int dT/T$, where T is defined in eq.(9), see Ref. [9] for details. Substituting eq.(11) into the partition function (1) and integrating over the $SU(2)$ field A we get:

$$
\mathcal{Z} = \sum_{n=0}^{+\infty} \frac{1}{n!} \int \prod_{i=1}^{n} \int \mathcal{D}\bar{x}^{(i)} \int \mathcal{D}\mathcal{A} \exp\{-S_{U(1)}(\mathcal{A}, j(s))\}\,,
\tag{12}
$$

where the abelian action is defined as follows:

$$
\begin{aligned}
\exp\{-S_{U(1)}(\mathcal{A}, j)\} &= \int \mathcal{D}A \exp\Big\{-\int d^4x \Big[\frac{1}{4}F_{\mu\nu}^2[A] - \lambda\big((A_\mu^1)^2 + (A_\mu^2)^2\big)\Big]\Big\} \\
&\quad \Delta_{ab}[A]\,\Delta_{FP}[A;\lambda]\,\delta\Big(j_\mu - \frac{g}{2\pi}\epsilon_{\mu\nu\alpha\beta}\partial_\nu f_{\alpha\beta}\big((A^{\Omega_s^{MaA}(A)})^3\big)\Big) \cdot \delta\Big(\mathcal{A}_\mu - (A_\mu^{\Omega_r^{MaA}(A)})^3\Big)\,,
\end{aligned}
\tag{13}
$$

here the limit $\lambda \rightarrow \infty$ is assumed. Thus the partition function for the $SU(2)$ gluodynamics in the MaA projection is rewritten as the partition function of some abelian theory which contains the gauge field and the monopoles.

Usually in the abelian projection the $U(1)$ gauge invariant quantities $(\mathcal{O})$ are considered. Below we derive the explicit expression for the $SU(2)$ invariant quantity $\tilde{\mathcal{O}}$ which corresponds to $\mathcal{O}$. The expectation value for the quantity $\mathcal{O}$ in the MaA gauge (18) is:

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$$< \mathcal{O} >_{MaA} = \frac{1}{\mathcal{Z}_{MaA}} \int \mathcal{D}A \, \exp\{-S(A) + \lambda R[A]\} \, \Delta_{FP}[A; \lambda] \, \mathcal{O}(A) \,. \tag{14}$$

Shifting the fields $U \to U^{\Omega}$ and integrating over Ω both in the nominator and in the denominator of expression (14) we get:

$$< \mathcal{O} >_{MaA} =< \tilde{\mathcal{O}} >, \quad \tilde{\mathcal{O}}(A) = \frac{\int \mathcal{D}\Omega \, \exp\{\lambda R[A^{\Omega}]\} \, \mathcal{O}(A^{\Omega})}{\int \mathcal{D}\Omega \, \exp\{\lambda R[A^{\Omega}]\}} \,, \tag{15}$$

$\tilde{\mathcal{O}}$ is the $SU(2)$ invariant operator. In the limit $\lambda \to +\infty$ we can use the saddle point method to calculate $\tilde{\mathcal{O}}$:

$$\tilde{\mathcal{O}}(A) = \frac{\sum\limits_{j=1}^{N(A)} Det^{\frac{1}{2}} M[A^{\Omega^{(j)}}] \, \mathcal{O}(A^{\Omega^{(j)}})}{\sum\limits_{k=1}^{N(A)} Det^{\frac{1}{2}} M[A^{\Omega^{(k)}}]} \,, \tag{16}$$

where $\Omega^{(j)}$ are the N–degenerate global maxima of the functional $R[A^{\Omega}]$ with respect to the regular gauge transformations Ω: $R[A^{\Omega^{(j)}}] = R[A^{\Omega^{(k)}}]$, $j, k = 1, \ldots, N$. In the case of non–degenerate global maximum ($N = 1$), we get $\tilde{\mathcal{O}}(A) = \mathcal{O}(A^{\Omega^{(1)}})$.

3   Numerical Results on the Lattice

On the lattice the partition function of the $SU(2)$ gauge theory is:

$$\mathcal{Z} = \int \mathcal{D}U \, \exp\{-S(U)\}, \quad S(U) = \frac{\beta}{2} \sum_{P} \text{Tr} \, (1 - U_P) \,, \tag{17}$$

$\mathcal{D}U$ is the Haar measure for the $SU(2)$ link fields U.

The lattice MaA projection is defined by the following maximization condition:

$$\max_{\{\Omega\}} R[U^{\Omega}], \quad R[U] = \sum_l \text{Tr}(U_l \, \sigma^3 \, U_l^+ \, \sigma^3) \,, \tag{18}$$

where the gauge transformed field U is $U^{\Omega}_{x,\mu} = \Omega_x^+ \, U_{x,\mu} \, \Omega_{x+\hat{\mu}}$.

Below we present the results of the numerical calculations of the monopole condensate on the lattice $10^3 \cdot 4$ with the anti–periodic boundary conditions. To see that we have the order parameter for the deconfinement phase transition it is convenient to study the probability distribution of the monopole creation operator Φ_{mon}, the details of the calculation are given in Ref. [5]. We calculate the expectation value $< \delta(\Phi - \Phi_{mon}(x)) >$. The physical meaning has the quantity $V(\Phi)$, which is defined as follows:

5

$$e^{-V(\Phi)} = < \delta(\Phi - \Phi_{mon}(x)) > , \qquad (19)$$

$V(\Phi)$ is an effective potential for the monopole field. It occurs, that in the confinement phase this potential is of the Higgs type; in the deconfinement phase $V(\Phi)$ has the minimum at the zero value of the field Φ. The position of the minimum of the potential, Φ_c, corresponds to the value of the monopole condensate. The quantity Φ_c strongly depends on the lattice volume, and we use the extrapolation procedure to get Φ_c^{inf} which corresponds to the infinite volume. In Fig. 1 we show the dependence of Φ_c^{inf} on the parameter β. At $\beta = \beta_c$ the deconfinement phase transition takes place. From this figure it is clearly seen that the monopole condensate exists ($\Phi_c^{inf} \neq 0$) in the confinement phase. This fact is in the agreement with the dual superconductor model of gluodynamics vacuum [1].
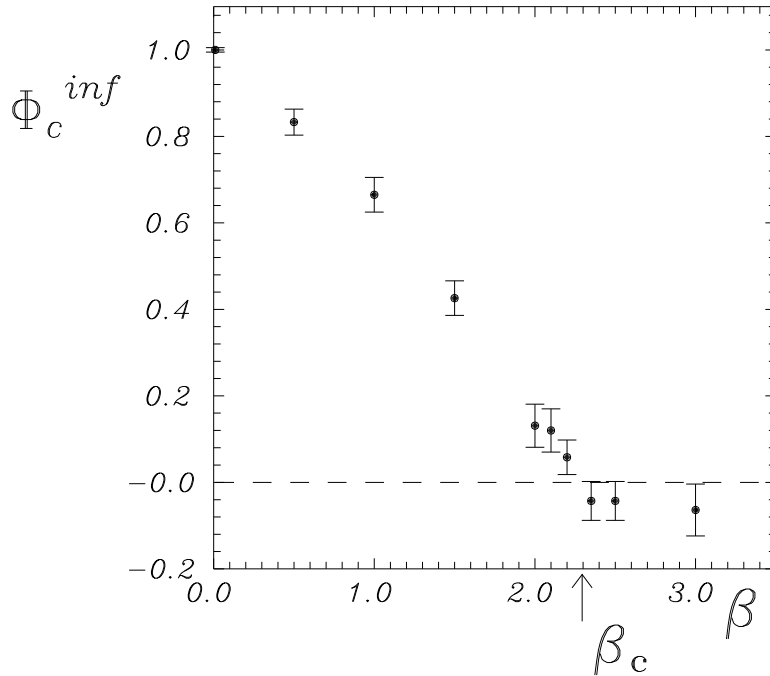

Fig. 1: The dependence of Φ_c^{inf} on β.

Acknowledgments

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